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► To cite this version:

Gaël Giraud. Walrasian Non-tatonnement with Incomplete and Imperfectly Competitive Markets. 2007. halshs-00155717

HAL Id: halshs-00155717

<https://shs.hal.science/halshs-00155717>

Submitted on 19 Jun 2007

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2007.21



CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE

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April 30, 2007

ABSTRACT.— Static competitive equilibria in economies with **incomplete markets** are generically constrained suboptimal. Allocations induced by strategic equilibria of imperfectly competitive markets are also generically inefficient. In both cases, there is scope for Pareto-improving amendments. In an extension of the limit-price process introduced in Giraud [20] to incomplete markets (with infinitely many uncertain states) populated by finitely many players, we show that these two inefficiency problems can be partially overcome when rephrased in a **non-tâtonnement process**. Traders are **myopic** and trade financial securities in continuous time, by sending limit-orders so as to select a portfolio that maximizes the first-order approximation of their expected indirect utility. We show that financial trade curves exist and converge to some second-best efficient rest-point unless some miscoordination stops the dynamics at some inefficient, but locally unstable, point.

KEYWORDS: incomplet markets, imperfect competition, second-best efficiency, non-tâtonnement.

RÉSUMÉ.— Les équilibre concurrentiels statiques d'une économie avec marchés incomplets sont génériquement sous-optimaux de second rang. Les allocations induites par les équilibres stratégiques de marchés imparfaitement concurrentiels sont également génériquement inefficients. Dans chaque cas, un espace est ouvert pour une amélioration Parétienne. Par une extension du processus de prix-limites introduit in Giraud [20] aux marchés incomplets (avec une infinité d'états incertains) peuplés d'un nombre fini de joueurs, on montre que ces deux problèmes d'inefficience peuvent être partiellement résolus une fois replacés dans un processus de non-tâtonnement. Les agents sont myopes et échangent des actifs financiers en temps continu, en envoyant des ordres de prix-limites de manière à sélectionner un portefeuille qui maximise l'approximation au premier ordre de leur utilité espérée indirecte. On montre que les courbes d'échanges financiers existent et convergent vers un point stationnaire efficient du second ordre à moins qu'une erreur de coordination n'interrompe la dynamique en un point inefficent, mais localement instable.

MOTS-CLEFS: marchés incomplets, concurrence imparfaite, efficience de second rang, non-tâtonnement.

JEL CLASSIFICATION: D11, D41, D50, E1

1 Introduction

The motivation of this paper is twofold. The first one consists in extending the framework of **Non-tâtonnement** to the setting of incomplete and imperfectly competitive markets. In most of the literature devoted to Non-tâtonnement, markets are taken to be complete and all transactions are assumed to take place under conditions of perfect competition.¹ The main result is usually of the following kind:

- The dynamics reduces to a differential inclusion.
- Every solution trade path converges to some Pareto point.

The main tool is Lyapounov's second method. Here, we aim at extending this approach to a stochastic set-up, having in mind the GEI model of incomplete markets populated by finitely many households where market transactions are still anonymously arranged but not necessarily under conditions of perfect competition. In fact, our underlying paradigm is taken from Giraud [19] and can be described as follows: Traders are myopic and trade financial securities in continuous time. These two features are captured by assuming that, at each time instant, traders behave so as to maximize their short-run utility, defined as the first-order linear approximation of their long-run utility function. Trades actually take place over time, and they are arranged according to Mertens' [24] limit price mechanism. This means that, at each time instant, traders are allowed to send limit orders to some central clearing house. The clearing house instantaneously calculates current market clearing prices and redistributes assets according to the executable orders.

The second motivation of our paper arises from two different bodies of literature. (a) On the one hand, the well-known failure of static competitive equilibria to be robustly second-best efficient when asset markets are incomplete (see Geanakoplos & Polemarchakis [16], Citanna, Kajii & Villanacci [7]). How to overcome this negative result ? Herings & Polemarchakis [23], for instance, suggest that the public authority should regulate prices. Giraud & Stahn [21] provide another answer in terms of strategic manipulation of prices: When markets are incomplete, imperfect competition can Pareto-dominate perfect competition. Here, we suggest still another answer in terms of continuous-time re-trading formalized as a price-quantity adjustment process. By doing so, we give a formal content to ??? celebrated theorem: Even when markets are missing, things might not be as "bad" as static GEI suggests provided people can retrade assets during a sufficiently long period of time until all the feasible opportunities of trade are exhausted. The battle over the viability of laissez-faire is almost as old as modern economics, and the GEI model provided a major step in freeing one's thinking from the automatic association of perfectly competitive equilibrium with Pareto optimality. At first glance, it might seem that our alternate dynamic approach of general equilibrium reverses the main lesson of the GEI model: Indeed, we show that, under quite general conditions, every perfectly competitive trade path will converge to some rest point that is nearly second-best efficient (according to a somewhat new definition of efficiency presented and discussed in section 2.4.). However, proponents of laissez faire should take our conclusions with some care: Indeed, since our dynamics is stochastic (as it should be in any uncertain environment), several definitions of probabilistic convergence are conceivable. The one adopted here is the weakest one, and a counter-example shows that it can hardly be strengthened. Thus, from our point of view, depending upon the kind of "insurance for efficiency" a society is looking for, there may still be a strong case for government intervention. (b) On the other hand, strategic market games populated by finitely many players generically induce Pareto-suboptimal allocations as Nash equilibria (see Dubey [9] *inter alia*). As is well-known, however, when the measure space of players become non-atomic, usually Cournot-Nash equilibria induce efficient outcomes due to the fact that negligible traders can no more manipulate prices. The question then arises, however, as to whether the set of (inefficient) strategic equilibrium outcomes with K players converges or not towards the set of (efficient) equilibria of the limit economy as $K \rightarrow +\infty$ (see Weyers [26], [27]) and the references

¹Aubin and Cellina [3], Champsaur and Cornet [6], Bottazzi [4], Giraud [20] among many others.

therein). Here, we show that, even when there are finitely many players interacting on the (incomplete) asset markets, there exist solution paths of our price-quantity adjustment process that do converge towards some second-best efficient outcome. The trajectories that fail to converge towards such limit-points are stopped, after a finite time, at some (constrained inefficient) state because of a coordination failure: At such a state, traders coordinated on the autarkic equilibrium of the corresponding short-run game. We prove, however, that such inefficient rest-points are locally unstable. Finally, we show that, in the two-commodity case, the generalized vector field induced by our imperfectly competitive dynamics always contains the vector field induced by the perfectly competitive case, and reduces to it as the number of agents grows to infinity. The size of the indeterminateness of solution trajectories can therefore be adopted as a measure of the imperfectness of competition (or, equivalently, of the illiquidity of markets).

A somehow analogous result is to be found in Ghosal and Morelli [18]. They show that Pareto-optimal allocations can be approximated when infinitely many rounds of retrading are allowed, and finitely many myopic agents who discount the future meet at each round. However, there, competition must be imperfect (while we cover the perfectly competitive case), markets are complete (while here they may be incomplete), time is discrete (while, here, it is continuous) and the trading paradigm is Shapley-Shubik (while, here, it is Mertens' [24] limit-price mechanism). Finally, Dubey *et alii* [10] allow a *continuum* of traders, who do not discount the future, to reopen the trading-posts of a Shapley-Shubik game before they consume their final allocations. The authors show that competitive equilibria can be achieved by allowing for arbitrarily (albeit finite) rounds of retrade. By contrast, here, we cover the case of finitely many players, and each of them is myopic in the sense that none of them does solve any intertemporal optimization problem. Convergence occurs, but towards merely second-best efficient allocations, and not towards competitive equilibria of the economy whose initial endowments equal the starting point of the dynamics.

2 The model

2.1 The incomplete markets economy

Uncertainty and assets²

We consider a pure-exchange economy \mathcal{E} with two periods $t = 0, 1$, uncertainty and incomplete markets of real securities. Uncertainty is captured through a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Events are revealed over time according to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$. No information is known at time $t = 0$ (hence $\mathcal{F}_0 = \{\emptyset, \Omega\}$), and all uncertainty is resolved at time $t = T$ (i.e., $\mathcal{F}_T = \mathcal{F}$). For simplicity, we assume that \mathbf{F} is the filtration generated by some stochastic process defined on Ω , with stationary increments.³ Since this covers the Brownian motion as a particular case, our setting contains the set-up usually studied in finance literature.

There are $J \geq 1$ long-lived securities. Each security j is a claim to some bundle $R_j \in L^2(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{R}_+^L)$.⁴ Let $R := (R_1, \dots, R_J)$. As soon as $\text{Span} R \neq L^2(\Omega, \mathcal{F}, \mathbf{P})$, markets are **incomplete**.⁵ A **portfolio** $z_i = (z_i^j)_j \in \mathbb{R}_+^J$ is a holding of each asset, and may include short sales.

vNM utilities

To each type of agent $i = 1, \dots, N$ is associated:

²Throughout this paper, $\mathbb{N}_K := \{1, \dots, K\}$.

³See, e.g., Arnold ([1] p. 546) for details.

⁴Hence, a security delivers only non-negative quantities of spot commodities.

⁵For an introduction to GEI, see Geanakoplos [15] ; for the peculiarities associated to incomplete financial markets with infinitely many states of the world, see Duffie [11] as well as the whole *JME* special issue (1996).

- (i) a known initial portfolio $z_i(0) \in \mathbb{R}_+^J$ of $J \geq 0$ real securities at time zero;
- (ii) a random (i.e., state-dependent) endowment $e_i(\cdot) \in L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ of $L \geq 1$ spot consumption commodities delivered at time T ;
- (iii) and vNM preferences over consumption bundles $x \in L^\infty$, where $x(\omega) \in \mathbb{R}_+^L$ describes time T consumption in state $\omega \in \Omega$. These preferences are assumed to be represented by a (Bernoulli) utility function $u_i : \mathbb{R}_+^L \rightarrow \mathbb{R}$ and a probability \mathbf{P} .⁶

The following assumption will be standing throughout the paper.

ASSUMPTION (C).

- (i) $\forall i, u_i$ is \mathcal{C}^1 , $\nabla u_i(\cdot) \gg 0$, u_i is strictly quasi-concave and verifies the boundary condition $u_i^{-1}(\lambda)$ is closed in \mathbb{R}^L .
- (ii) $(\Omega, \mathcal{F}) \subset (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and \mathbf{P} is Borel.
- (iii) $\forall z \in \tau, \forall i,$

$$\omega \mapsto u_i[e_i(\omega) + z_i \cdot R(\omega)] \text{ is in } L^1(\Omega, \mathcal{F}, \mathbf{P}).$$

- (iv) $\forall K \subset \mathbb{R}_+^L$ compact (nonvoid), $\exists g \in L^1(\Omega)_+$ s.t.

$$\left| \frac{\partial}{\partial x} u_i(e_i(\omega) + z_i(\omega)) \right| \leq g(\omega), \quad \forall z \in \tau, \text{ a.e. } \omega \in \Omega.$$

Example. When $|\Omega| < +\infty$, assumptions (ii-iv) are automatically fulfilled.

For all i , $V_i : \mathbb{R}_+^J \rightarrow \mathbb{R}$ denotes the indirect utility expected by agent i from a portfolio $z_i \in \mathbb{R}_+^J$:

$$V_i(z_i) := E^{\mathbf{P}}[u_i(e_i + z_i \cdot R)]. \quad (1)$$

A **financial allocation** $z = (z_i)_i$ is feasible if:

$$\sum_i z_i^j = \sum_i z_i^j(0),$$

and

$$\forall i, e_i + z_i \cdot R \geq 0 \text{ a.s.}$$

Feasibility requires that all promises be honored at T : Trades after T , in spot commodities, start with individually feasible consumption bundles $e_i + z_i \cdot R \geq 0$, at least almost surely. This means that, before T , agents are allowed to go arbitrarily short $z_i^j < 0$ in any asset j provided they ultimately keep their promises (by reducing their initial holding at time T by $-z_i \cdot R$).⁷ Allowing short sales violates the standard physical feasibility conditions of strategic market games (SMG).⁸ Thus, at first glance, it does not seem to be compatible with the SMG approach. Nevertheless, we shall see that the possibility of short sales does not interfere with existence of trade curves. We denote by $\tau \subset (\mathbb{R}_+^J)^N$ the subset of **feasible** financial allocations.

The stochastic short-run economy

For every state ω , the short-run economy $T_{z,\omega}\mathcal{E}$ can now be defined, at $z \in \tau$, as the linear economy:

$$T_{z,\omega}\mathcal{E} := (\nabla V_i(z_i), z_i), \quad (2)$$

where household's i short-run trade space is

⁶ \mathbf{P} could be made type-dependent without altering the analysis.

⁷See Geanakoplos & Shubik [17] for a similar treatment of short sales in a static GEI environment.

⁸See Giraud [19] for an introduction to strategic market games.

$$\{\dot{z}_i \in \mathbb{R}^J : (z_i + \dot{z}_i) \cdot R \geq -e_i \text{ a.s. } \}.$$

In words, her instantaneous initial endowment in securities is 0; her short-run preferences are given by the first-order approximation of her expected utility over assets. In other words, i myopic valuation of the infinitesimal trade \dot{z}_i is:

$$\nabla V_i(z_i) \cdot \dot{z}_i, \quad (3)$$

where $\nabla V_i(z_i) = (\frac{\partial}{\partial x_i^j} V_i(z_i))_j$ is the gradient of the expected indirect utility with respect to assets $j = 1, \dots, J$. Assumptions (C)(iii) and (iv) guarantee that one can differentiate with respect to z_i under the expectation operator of (1), so that (3) is well-defined.

2.2 The local game

To $T_{z,\omega,s}\mathcal{E}$ is associated a local non-cooperative game $G[T_{z,\omega,s}\mathcal{E}]$ defined as in Weyers [26], [27] with the help of Mertens' [24] limit-price mechanism.

Actions

An action is a J -tuple of correspondence offers $S_i(p)$ from the space of prices \mathbb{R}_+^J to \mathbb{R}_+ , which are homogeneous of degree zero, usc⁹, convex valued, and non-decreasing in its own price (in the sense that every measurable selection of S_j is non-decreasing as a function of p_j).

These correspondences represent continuous sums of limit-price orders (that depend upon the whole stream of prices).

The outcome

When actions of the above form are sent to the market, the limit-price mechanism instantaneously finds a final allocation and a price (up to some positive scalar) as follows. If the actions offer no mutually beneficial possibilities for trade, the final allocation is the same as the initial allocation and there may be multiple price systems that support this. If there are possibilities of trade, the mechanism finds the unique final allocation, and a price system which equilibrate the supply of all goods. Formally, the mechanism finds p such that, for every pair j, k of securities,

$$p_j \sum_i q_j^i = p_k \sum_i q_k^i \quad \text{for some } q_j^i \in S_j^i(p), q_k^i \in S_k^i(p).$$

However, if the market-clearing price is a limit-price, there may be multiple market-clearing allocations. The exact amount that will be sold is determined by the proportional rule (see Mertens [24], Weyers [27], Giraud [20]).

Imperfectness of competition is measured through the number of clones k of type $i = 1, \dots, N$. When $k = 1$, imperfectness is maximal. As $k \rightarrow +\infty$, the weight of each individual becomes negligible. At the limit, we consider the case where each type i is represented by a non-atomic space $([0, 1], \mathcal{B}[0, 1], \lambda)$ of identical traders.¹⁰ Thus, in each case, the space of players is some measure space (I, \mathcal{I}, ν) where ν is either the counting (when I is finite) or the Lebesgue (when I is a continuum) measure.¹¹

Let \mathcal{K} denote the dimension of some Euclidean space containing each player's i pure action set $S(i)$. An *action selection* is a function $f : I \rightarrow \mathbb{R}^{\mathcal{K}}$, each coordinate of which is Lebesgue integrable, such that, for ν -a.e. $i \in I$, $f(i) \in S(i)$. Let F_S denote the space of all strategy selections, and, for any $f \in F_S$, let $s(f) := \int_I f(h) d\nu(h)$, $\mathcal{S} := \{s(f) : f \in F_S\}$, and $\varphi(s(f)) \in \tau$ (resp. $\varphi_i(x, s(f))$) be the feasible outcome induced by f according to the rules

⁹Upper semi-continuous, i.e. with a closed graph and a bounded range.

¹⁰For simplicity, we take the real interval $[0, 1]$ for each type, together with the Borel tribe $\mathcal{B}[0, 1]$ and the restriction of the Lebesgue measure λ . This involves no real loss of generality since every complete probability space is measurably isomorphic to $([0, 1], \mathcal{B}[0, 1], \lambda)$.

¹¹There would be no difficulty in allowing ν to be of the "mixed" form: $\nu = \lambda + \mu$, where μ is purely atomic.

of the limit-price mechanism¹² (resp. the outcome received by i when she plays $x \in S(i)$ while the population plays f). A Nash equilibrium (NE, sometimes called “Cournot-Nash equilibrium” when I is non-atomic) is an action selection \hat{f} such that, for ν -a.e. i ,

$$\nabla V_i(z_i, s) \cdot \varphi_i(\hat{f}) \geq \nabla V_i(z_i, s) \cdot \varphi_i(x, s(\hat{f})) \quad \forall x \in S(i).$$

2.3 The dynamics

The price-quantity adjustment dynamics is defined as follows: At each time instant $t \geq 0$, if the state is $z(t) \in \tau$, players play a NE in the local game $G[T_{z(t)}\mathcal{E}]$ associated to $T_{z(t)}\mathcal{E}$. The set of outcomes in $T_{z(t)}\mathcal{E}$ induced by these NE provides the set of directions in which the long-run economy moves in τ between t and $t + dt$. The current price $p(t)$ is one of the prices induced by the NE of $G[T_{z(t)}\mathcal{E}]$. We denote by $\pi[\bar{s}(z(t))]$ the set of normalized prices in the unit sphere S^{J-1} induced by the NE \bar{s} of $G[T_{z(t)}\mathcal{E}]$. When I is non-atomic, we know from Giraud [20] that there is a unique NE in $G[T_{z(t)}\mathcal{E}]$, which is also a dominant strategy equilibrium, and where players play truthfully, in the sense that they send to the clearing-house the limit-price orders that mimic their linear short-run preferences. When I is finite, there is still existence of NE, but no more uniqueness.

LEMMA 1.— *When I is finite, every local game $G[T_{z(t)}\mathcal{E}]$ admits no-trade as NE. However, every quasi-outcome¹³ of $T_{z(t)}\mathcal{E}$ can be obtained as a NE of $G[T_{z(t)}\mathcal{E}]$.*

Proof. That no-trade is always a NE is well-known and easy: If everybody but i bids 0 in the local game $G[T_{z(t)}\mathcal{E}]$, bidding 0 is a best-reply for i .

It follows from Mertens’s [24] that $T_{z(t)}\mathcal{E}$ admits a quasi-outcome since, according to (C), every player’s linear short-run indirect utility is (weakly) increasing in each security (paying non-negative returns). Take such a quasi-outcome (\dot{z}, π) . Decompose each player i ’s net trade \dot{z}_i into a sum of vectors

$$\dot{z}_i = \sum w_\ell$$

having each a single negative and positive coordinate, and having zero value under π . Construct an action for player i consisting in sending each w_ℓ together with π , as a limit-price order. The action profile obtained by repeating this operation for each player induces the original quasi-outcome as final outcome, and is a NE because under any unilateral deviation of individual i , i ’s net trade will be the opposite of the sum of all other net trades, each of whom has a non-negative value under π .

Hence the dynamics in the ... space τ is given by the following differential inclusion:

$$\begin{aligned} j(t) &\in \phi(\bar{s}(z(t))), \quad t \geq 0, \\ j(0) &= \omega, \end{aligned}$$

while in the price sphere S^{J-1} , it is given by

$$p(t) \in \pi[\bar{s}(z(t))], \quad t \geq 0.$$

□

¹²See the Appendix.

¹³Cf. Giraud [20], see also Mertens ([24] section 8.2 p. 526) for a definition.

2.4 How to define efficiency ?

Already in the deterministic case (i.e., when Ω reduces to a singleton and when competition is perfect (i.e. I is non atomic)), the underlying perfectly competitive dynamical system is discontinuous (Giraud [20]). Thus, we use the following concept of solution (both in the perfectly as well as in the imperfectly competitive cases):

DEFINITION 1 (Filippov [14]) A **Filippov solution** of the differential inclusion

$$\dot{z}(t, \omega) \in \varphi(\bar{s}(z(t, \omega), t, \omega)) \quad t \geq 0, \quad (4)$$

$$z(0) \in (\mathbb{R}_{++}^J)^N \quad (5)$$

is an absolutely continuous trajectory $\phi : [a, b) \times \Omega \rightarrow \tau$ such that, for a.e. $t \in [a, b) \subset \mathbb{R}$, and a.e. $\omega \in \Omega$,

$$\dot{\phi}(t, \omega) \in \Phi(\phi(t)) := \cap_{\varepsilon > 0} \cap_{A \in \mathcal{N}} \overline{\text{co}}\{y \mid d(y, \varphi(\bar{s}(z(t, \omega), t, \omega))) < \varepsilon, y \notin A\}. \quad (6)$$

Observe that, in the definition of $\Phi(\cdot)$, only the limits in the allocation space are taken into account: t and ω (hence, also the information publicly available at time t according to ω) are fixed. Notice also that Filippov's convexification method, applied to the differential equation with discontinuous right-hand side (4) yields in (6) a standard differential inclusion associated with a non-empty and convex valued correspondence.

LEMMA 2.— *The right-hand side of (6) is an usc and locally bounded correspondence with non-empty and convex values.*

Proof. Convexity is obvious from the definition. For each $z \in \tau$ and a.e. ω , the set

$$\Phi(z) := \cap_{\varepsilon > 0} \cap_{A \in \mathcal{N}} \overline{\text{co}}\{y \mid d(y, \varphi(\bar{s}(z(t, \omega), t, \omega))) < \varepsilon, y \notin A\}$$

is closed. For every neighborhood U^{14} of z in τ , $\Phi(V)$ is bounded. Indeed, being feasible in the short-run economy $T_{z, \omega, t} \mathcal{E}$, $\varphi(\bar{s}(z(t, \omega), t, \omega))$ is uniformly bounded by $\bar{z}(0)$. Finally, the graph of Φ is the closure of the graph of the set-valued map $\varphi(\bar{s}(\cdot))$, and is therefore closed. Upper semi-continuity then follows, e.g., from Filippov ([14], Lemmata 14 and 15 p. 66).

It remains to prove non-emptiness, i.e., essentially to show that the possibility of arbitrary sort sales does not preclude the existence of a dominant strategy (in the perfectly competitive case) or of a NE (in the imperfectly competitive case) in every local game $G[T_{z, \omega, t} \mathcal{E}]$. For this purpose, first consider the auxiliary truncated local game $G^K[T_{z, \omega, t} \mathcal{E}]$ attached to each tangent economy $T_{z, \omega, t} \mathcal{E}$, obtained from $G[T_{z, \omega, t} \mathcal{E}]$ by restricting each player's i action to stay in K , where K is some arbitrary rectangle of \mathbb{R}^J centered on 0. Formally, for every asset j , every price q and every player i , one must have

$$S_j^i(q) \subset K.$$

Here, existence follows from standard arguments (see, e.g., Weyers [27] and Giraud [20]). Now, let K_n be a sequence of successively larger rectangles, whose radius tends to infinity. For each n , there must be some equilibrium (either Nash or dominant according to the set-up). Since the period T -outcome $x_n = (x_n^i)$ induced by this very equilibrium must be feasible, and since the period T feasible set is compact (because consumption sets are bounded below), it follows, by passing to a convergent subsequences, that one can find some limit of the sequence of equilibria as $n \rightarrow \infty$. A standard argument then enables to show that this limit is itself an equilibrium.¹⁵

□

A **trade curve** is a solution of (4) in the sense of Filippov. Filippov ([14], p. 85)'s existence theorem for (6) holds for non-autonomous systems, so that existence goes through

¹⁴ U is called a neighborhood of a set A if U contains an open set that contains A .

¹⁵This line of reasoning originates in Geanakoplos & Polemarchakis [16].

as in the deterministic case studied in Giraud [20], both in the perfectly and imperfectly competitive cases. Hence,

PROPOSITION 1. *Under (C), every economy \mathcal{E} admits a trade curve.*

Remark 1. In the perfectly competitive case (i.e., when I is non-atomic), this existence result should be contrasted with Hart’s [22] example of non-existence of static GEI equilibria, and compared with the (generic both in endowments and asset structure) existence result for GEI economies with real assets (Duffie & Shafer [13]).

The main issue is whether solution trajectories of our stochastic dynamical process will converge towards anything akin to the Pareto set. Notice that Lyapounov’s classical theorem is stated for autonomous, deterministic systems. A step in the direction of non-autonomous systems was made by Bottazzi [4]. However, here, the main difficulty is, of course, the fact that the underlying system is not deterministic.

First, we need to define efficiency in a manner suited to our environment. Following a hint in Diamond (1967) about constrained efficiency, and successive refinements by Diamond (1980), Loong and Zeckhauser (1982), Newbert and Stiglitz (1982), Stiglitz (1982), Greenwald and Stiglitz (1984), a reasonable definition of GEI constrained efficiency became available in Geanakoplos and Polemarchakis (1986). The main intuitive idea was that, in discussing efficiency of GEI equilibrium, it is necessary to restrict the set of alternative allocations to what is feasible, given the constraints under which the market is operating, in addition to the usual physical feasibility conditions. Thus, the definition adopted in Geanakoplos and Polemarchakis (1986) (and followed in numerous subsequent papers already cited) was the following. Suppose that a planner can intervene in a two-period GEI economy only at date zero before nature moves, inducing agents to hold alternative portfolios of existing assets and consumption goods. Subsequently, in each state in time 1, a competitive equilibrium in spot commodities takes place and markets clear. If there is such an intervention that is Pareto improving, then the initial GEI equilibrium was not constrained efficient. Given our dynamic perspective, this definition has to be amended in two directions. First, since we assume that “trade takes time”, it makes no sense to allow the planner to induce a “jump” in the portfolio space: an intervention of the planner must now be described by means of a Pareto improving path. This implies that we should switch from a global point of view to a local one: instead of considering any alternative feasible reallocation of portfolios, one should focus on “beginnings” of paths, i.e. on infinitesimal changes of allocations, among which one should look at possibly Pareto improving ones.

On the other hand, it hardly makes sense to take for granted that a new, static competitive equilibrium will take place in spot commodity markets at date 1. Trades in spot commodities should evidently be described in turn by means of paths.

Fix some some state ω , and some short-run economy $T_{z,\omega,t}\mathcal{E}$.

DEFINITION 2.

- (i) A feasible trade \dot{z} is *strict* if, for each i , either $\dot{z}_i \geq 0$ or $\nabla V_i(z_i(t), \omega, t) \cdot \dot{z}_i > 0$.
- (ii) An allocation $z \in \tau$ is **constrained infinitesimally efficient at time $t \in [0, T]$** if there does not exist any path ϕ with $\phi(a) = z$, $\phi'(a)$ is a strict trade in $T_{\phi(a),\omega,t}\mathcal{E}$, and

$$\nabla V_i(z_i(t), \omega, t) \cdot \phi'_i(a) \geq 0 \quad \forall i$$

with at least one strict inequality.

- (ii) Given $\eta > 0$, an allocation z is **constrained η -infinitesimally efficient at time $t \in [0, T]$** if there does not exist any path ϕ with $\phi(a) = z$, $\phi'(a)$ is a strict trade in $T_{\phi(a),\omega,t}\mathcal{E}$, and

$$\nabla V_i(z_i(t), \omega, t) \cdot \phi'_i(a) \geq \eta \quad \forall i,$$

with at least one strict inequality.

Of course, constrained η -infinitesimal efficiency at time $t < T$ implies constrained η -infinitesimal efficiency at $t' \in [t, T]$. For $\eta > 0$, let $\theta_\eta(\omega, t)$ be the subset of constrained η -infinitesimally optimal allocations.

Remark 2. When markets are complete (e.g., when Ω reduces to a singleton), constrained efficiency reduces to standard Pareto-optimality. Notice that second-best efficiency in the sense of Definition 2 departs from the one used in Geanakoplos & Polemarchakis [16] in as much it does not rely upon some second-period equilibrium. As a consequence, even if a constrained efficient allocation in assets is reached in finite time, there may be a need for reopening spot markets after time T . One way to understand this definition (and remain close to the spirit of Geanakoplos & Polemarchakis [16]) could consist in supposing that a single spot commodity can be traded after period T . But this would destroy a large part of the interest of our approach, at least in the perfectly competitive case, since static GEI equilibria are already second-best efficient in economies with a unique spot commodity.

2.5 The results

We shall need a weak restriction, expressed in terms of inexecutable orders.¹⁶

DEFINITION 3. (i) A linear economy $\mathcal{L} = (I, \mathcal{I}, \mu, b, e)$ is “*weakly reducible*” if there exists a partition $A \cup B = \mathbb{N}_J$ such that for each “agent” i , either $b_b^i = 0 \ \forall b \in B$, or $e_a^i = 0 \ \forall a \in A$, and there exists some triple (i_0, b, a) with $e_b^{i_0} > 0$, $b_b^{i_0} = 0$ and $b_a^{i_0} > 0$.

(ii) \mathcal{L} is *weakly irreducible* if it is not weakly reducible, i.e., if it admits no inexecutable order. In such an economy, the orders to sell item $b \in B$ are clearly inexecutable.

A trade curve $\varphi(\cdot)$ is said to exhibit no inexecutable order if each of the short-run economy $T_z\mathcal{E}$ attached to some point z crossed by $\varphi(\cdot)$ admits no inexecutable order.

Several concepts of convergence are available in our stochastic setting. Here, we use one of the weakest possible criteria. A sequence $(X_n)_n$ of random variables **converges** to a random variable (r.v. in the sequel) X **in probability**, i.e., $\mathbf{P} \lim_{n \rightarrow +\infty} X_n = X$, if:

$$\lim_n \mathbf{P} \left[\omega : |X_n(\omega) - X(\omega)| > \varepsilon \right] = 0 \quad \text{for all } \varepsilon > 0.$$

Our main result is the following:

THEOREM 1.— *Under (C), in the perfectly competitive case, $\forall \eta > 0, \exists T$ sufficiently large such that $\forall \varepsilon > 0$, every trade curve $\varphi(\cdot)$ with no inexecutable order verifies:*

$$\mathbf{P} [d(\varphi_T | \theta_\eta(T)) \geq \varepsilon] < \varepsilon. \quad (7)$$

In words: if households have sufficiently many time to trade, and if competition is perfect, the economy converges in probability to some point that is arbitrarily close to being constrained efficient. How close the economy will end up to some efficient allocation of assets depends upon the size T of the time horizon. The larger T is, the smaller η can be chosen, hence the closer the limit-point will be to the random subset of constrained efficient states. Actually, the proof shows that, for $T = +\infty$, the corresponding z is globally asymptotically stable for the metric of convergence in probability.

When competition is imperfect, almost the same conclusion holds with the following proviso: At every time instant t , it may be the case that players mistakenly coordinate on the autarkic NE of $G[T_{z,\omega,t}\mathcal{E}]$, in which case z is a rest-point of the trajectory. But the next Proposition says that such a z must be locally unstable.

DEFINITION 4. A random allocation $x : \Omega \rightarrow \tau$ is said to be *locally unstable* for (6) if, $\forall \varepsilon > 0$, and a.e. ω , there exists a neighborhood, $C_\omega \subset \tau$, of x , of radius less than ε and such

¹⁶See Giraud [20] for a discussion of this assumption.

that, from each $y \in C(\omega)$ there exists a path $\varphi : [a, b) \rightarrow \tau$, verifying (6), with $\varphi(a) = y$ and leaving C_ω in finite time, i.e., such that $\varphi(t) \notin C_\omega$ for some $a < t < b$.

An alternative local instability concept would go as follows: The random variable x is weakly locally stable under (6) if $\forall \varepsilon > 0, \exists$ a random neighborhood, C , of A , such that:

$$\triangleright \mathbf{P}[d(C|A) \geq \varepsilon] < \varepsilon,$$

and, for a.e. ω , and every $y \in C(\omega)$, all the solution paths of (6) starting from y remain in $C(\omega)$.

Every point failing to be locally stable (in this second sense) would be declared locally unstable. In this alternative case, however, $C(\omega)$ can be possibly far away from $\{x\}$ for an ω -set of small (but not zero) probability. The choice of $C(\cdot)$ in this second definition is equivalent to choosing a neighborhood ε -close to $\{x\}$ with respect to the metric

$$\rho(X, Y) = \inf\{\varepsilon \geq 0 \mid \mathbf{P}[|X - Y| \geq \varepsilon] \leq \varepsilon\},$$

which completely metrizes the topology of convergence in probability over the space of \mathbb{R}^n -valued random variables. On the other hand, as soon as, for every such neighborhood, there exists at least *one* point from which an “outside pointing” path can start, then x is no more stable. By contrast, when checking the local instability of $\{x\}$ in the sense of Definition 4., one asks for a neighborhood C that remains close to $\{x\}$ for a.e. ω . There are much less neighborhoods verifying this criterion than closedness in probability. On the other hand, however, it is asked in Definition 4 that, from *every* point y in such a neighborhood there be at least one starting path that leaves the neighborhood in finite time.

PROPOSITION 2.— *In the imperfectly competitive case, trade curves with no in-executable order failing to satisfy (7) in Theorem 1 stop at some locally unstable random allocation.*

Remark 3. Of course, in the deterministic case, our notion of local instability reduces to the standard one, so that we can conclude, as a by-product, that, when $\Omega = \{\omega\}$, every inefficient allocation is locally unstable (in the usual sense). Notice, however, that even in the deterministic case, no single Pareto optimal point x can be locally asymptotically stable because of the indeterminacy of Pareto points. (Remember that, in the standard differentiable setting, the set of optima is a $(N - 1)$ -dimensional submanifold of the feasible set.)

Remark 4. Duffie & Huang [12] consider a problem that is close (but not identical) to the one studied here. They prove that continuous-time retrading enables to mimic every (complete markets) Arrow-Debreu equilibrium by means of an (incomplete markets) Radner equilibrium. The main difference with our approach is that, here, agents are **myopic**, and we get only a probabilistic convergence towards approximately second-best efficiency. There, agents are **far-sighted** in the sense that they are assumed have rational expectations, and to be able to solve an intertemporal optimization programme. As a consequence, they get exact first-best efficiency.

We end this section with a result concerning economies with $J = 2$. A “generalized random vector field”, here, is a set-valued map: $\nu : \tau \times \Omega \rightarrow T\tau$. If ν and Ω are single-valued, one recovers a standard vector field. For each $k = 1, 2, \dots$, let us denote by ν_k the generalized random vector field associated to the dynamics (6) with imperfect competition, when each type i is represented by k clones. Finally, ν_∞ denotes the vector field obtained in the perfectly competitive case. Given two generalized vector fields ν_1 and ν_2 , we write $\nu_1 \subset \nu_2$ with the meaning that, on each point z in $\text{int}\tau$, $\nu_1(z, \omega) \subset \nu_2(z, \omega)$, almost surely. Finally, $\mathbf{0}$ denotes the trivial vector field that associates the 0 vector to each point in τ (i.e., the trivial section of the tangent bundle of τ).

PROPOSITION 3.— *Under (C), when $J = 2$ and Ω is finite, then, for every $k \geq 1$,*

$$\nu_\infty \subset \nu_{k+1} \subset \nu_k.$$

Moreover, $\cup_k \nu_k = \nu_\infty \cup \mathbf{0}$.

Roughly speaking, this means that the perfectly competitive trajectory is always included in the neighborhood of imperfectly competitive trajectories, and that this neighborhood shrinks to the perfectly competitive trajectory (or to the collection of competitive trajectories stopped at some locally unstable point before having reached θ) when competition becomes perfect. As a consequence, indeterminacy is a characteristic of imperfect competition. This raises the question of how indeterminate the set of imperfectly competitive trajectories are. The next, and last, result gives a partial answer in the peculiar 2×2 case. It should be contrasted with the parallel result obtained in Giraud [20] for the perfectly competitive case, where the trade curve is shown to be globally unique for every interior starting point.

PROPOSITION 4.— *When $N = J = 2$, for every interior starting point $z(0)$, i.e., such that $z_i(0) \gg 0$ for every i , then every interim Pareto efficient allocation that is individually rational with respect to can be approximated by means of some trade curve.*

3 Proofs.

3.1 Proof of Theorem 1

In a first step, we provide the essentials of the proof of a more general result (Theorem 2) that is interesting in its own right, and could be applied to alternative economic contexts. In a second step, we shortly apply Theorem 2 to our context in order to get Theorem 1 (i) as a corollary.

Step 1. We consider a multivalued random semi-flow¹⁷ whose ingredients are the following:

- (i) $(\Omega, \mathcal{F}, \mathbf{P})$ is the probability space modelling uncertainty ;
- (ii) $z_t : \Omega \rightarrow \Omega$ a measure preserving group of transformations ρ in Ω such that $(t, \omega) \mapsto \rho_t \omega$ is measurable, and

$$\begin{aligned} \rho_{t+s} &= \rho_t \circ \rho_s = \rho_s \circ \rho_t, \quad t, s \in \mathbb{R}_+ \\ \rho_0 &= \text{Id}_\Omega. \end{aligned}$$

- (iii) A non-empty, compact feasible set $\tau \in \mathbb{R}^L$.
- (iv) A set-valued map $G : \mathbb{R}_+ \times \Omega \times \tau \mapsto K(\tau)$ that constitutes a multivalued random dynamical system, i.e., is measurable and verifies:

- a) $G(0, \omega) = \text{Id}_\tau$;
- b) $G(t + s, \omega)x = G(t, \rho_s \omega)G(s, \omega)x$ for all $t, s \in \mathbb{R}_+, x \in \tau, \omega \in \Omega$.

Note that a multi-valued random semi-flow reduces to a deterministic differential inclusion in case the random noise ω is absent.

¹⁷This is the multi-valued generalization of random dynamical systems. See Arnold [1] for a classical textbook on random dynamical systems, and Caraballo *et alii* [5] for the asymptotic studies of their set-valued extensions.

Since \mathbf{F} is generated by some stochastic process with stationary increment, the unfolding of information as formalized with the help of the filtration \mathbb{F} can be rephrased as a \mathbf{P} -preserving group of transformations ρ_t over Ω (see Arnold [1] p. 546 for details). From now on, we shall do so. Next, the set of constrained efficient allocations is random. Let us denote by $\mathcal{K}(\mathbb{R}^{NJ})$ the set of non-empty compact subsets of the portfolio allocation space \mathbb{R}^{NJ} .

DEFINITION.— A *random compact [resp. closed] set* D is a measurable map

$$D : \Omega \rightarrow \mathcal{K}(\mathbb{R}^{NJ})$$

[resp. from Ω into the family of non-empty closed subsets of \mathbb{R}^{NJ}] in the sense that, given $x \in \tau$, the map $\Omega \ni \omega \mapsto d(x, D(\omega))$ is measurable.¹⁸

Since \mathcal{F} is complete, the property of being a random compact [resp. closed] set is equivalent to graph D being $\mathcal{F} \otimes \mathcal{B}^{NJ}$ -measurable and $D(\omega)$ being compact [resp. closed].¹⁹ Recall that, if A and B are non-empty closed subsets of \mathbb{R}^d , the Hausdorff semi-metric $d(A|B)$ is defined by:

$$d(A|B) := \sup_{x \in A} d(x, B)$$

while

$$d_H(A, B) := d(A|B) + d(B|A)$$

denotes the Hausdorff metric, which makes $\mathcal{K}(\mathbb{R}^d)$ a Polish space. We refer to Arnold [1] for a proof of the following facts:²⁰

- LEMMA 3.1.1.— (i) If M is a random closed set in \mathbb{R}^d , then so is $\overline{M^c}$ the closure of the complement $M^c := \mathbb{R}^d \setminus M$.
- (ii) If M is a random set, \overline{M} is a closed random set.
- (iii) If M is a random closed set, $\text{int}M$, the interior of M is a random open set (i.e., its complement $(\text{int}M)^c$ is a random closed set).
- (iv) If $(M_n)_{n \in \mathbb{N}}$ is a sequence of random compact sets with non-void intersection, then $\cap_n M_n$ is a random compact set.
- (v) If $(M_n)_n$ is a sequence of random compact sets, and if $M := \overline{\cup_n M_n}$ is compact, then M is a random compact set.
- (vi) If M is a random compact set and $F : \Omega \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a set-valued map such that $F(\omega, \cdot)$ is upper-semi-continuous with compact values, and $F(\cdot, x)$ is measurable for all x , then $\omega \mapsto F(\omega, M(\omega))$ is a random compact set.
- (vii) A function $\omega \mapsto M(\omega)$ taking values in the non-empty closed subsets of \mathbb{R}^d is a random closed set iff there exists a sequence $(X_n)_{n \in \mathbb{N}}$ of measurable maps $X_n : \Omega \rightarrow \mathbb{R}^d$ such that

$$M(\omega) = \overline{\{X_n(\omega) : n \in \mathbb{N}\}} \quad \text{for all } \omega \in \Omega.$$

Note that, if X is a random variable and A and B are random closed sets, the mappings $d(X|B)$ and $d(A|B)$ are random variables since, by Lemma 3.1.1 (vii) the sup and inf over an uncountable set can be replaced by the one over an exhausting countable set. In particular, the sets: $\{\omega \mid X(\omega) \in B(\omega)\} = \{d(X(\omega), B(\omega)) = 0\}$ and $\{A \subset B\} = \{d(A|B) = 0\}$ are measurable.

¹⁸Where $d(x, M) := \inf_{y \in M} \|x - y\|$.

¹⁹Where \mathcal{B}^d denotes the Borel tribe over \mathbb{R}^d .

²⁰Actually, (vi) is proven there only when F is a function, but the proof of the slightly more general case is the same.

Since we want to prove convergence in probability of a multivalued random semi-flow (to which our dynamics will shortly reduce), we need to extend the notion of stochastic local stability to random sets.

DEFINITION.— A random set A is **stable** under G if $\forall \varepsilon > 0, \exists$ a random neighborhood, C , of A , such that:

$$\begin{aligned} \triangleright \mathbf{P}[d(C|A) \geq \varepsilon] &< \varepsilon. \\ \triangleright G(t, \omega)C(\omega) &\subset C(\rho_t \omega) \quad \forall t \geq 0. \end{aligned}$$

Observe that this is but the set-valued and global analog of the notion of stability alluded to *supra* as an alternative to Definition 4. The second property is referred to as meaning that C is *forward invariant* under G . Notice also that, since $A \subset C$, there is no need to refer to the Hausdorff metric in this definition: Actually, $d_H(A, C) = d(A|C)$.

The following is the set-valued analog of Lemma 4.3. in Arnold & Schmalfuss [2].

LEMMA 3.1.2.— Suppose that C is a forward invariant random compact set under G . Then,

$$A(\omega) := \cap_{t \geq 0} G(t, \theta_{-t}\omega)C(\theta_{-t}\omega) \quad (8)$$

is an invariant random compact set, i.e.,

$$G(t, \omega)A(\omega) = A(\theta_t \omega) \quad \forall t \in \mathbb{R}_+.$$

Moreover, A is attracting C in the “pull-back sense”, i.e.,

$$\lim_{t \rightarrow +\infty} d(G(t, \theta_{-t}\omega)C(\theta_{-t}\omega)|A(\omega)) = 0 \quad \forall \omega. \quad (9)$$

Note that the last convergence result implies that A is attracting C in the sense of convergence in probability:

$$\mathbf{P} - \lim_{t \rightarrow +\infty} d(G(t, \theta_{-t}\omega)C(\theta_{-t}\omega)|A(\omega)) = 0.$$

Proof. For each state ω , $A(\omega)$ is the intersection of a decreasing sequence of non-empty compact sets contained in $C(\omega)$, hence is non-void, compact and included in $C(\omega)$. Using the elementary fact that, if $(C_n)_n$ is a decreasing sequence of compact sets and if F is a set-valued map, then for every finite family of indices $N \subset \mathbb{N}$,

$$\cap_{n \in N} F(C_n) \subset F(\cap_{n \in N} C_n),$$

and applying the upper semi-continuity of F , one gets by taking the limit:

$$\cap_{n \in \mathbb{N}} F(C_n) \subset F(\cap_{n \in \mathbb{N}} C_n).$$

Now, the cocycle property and the monotonicity (for inclusion) of $(G(t, \theta_{-t}\omega), C(\theta_{-t}\omega))$, one obtains for any $T \in \mathbb{R}_+$:

$$\begin{aligned} G(T, \omega)A(\omega) &\supset \cap_{t \geq 0} G(T, \omega, G(t, \theta_{-t}\omega), C(\theta_t\omega), C(\theta_t\omega)) \\ &= \cap_{t \geq T} G(t, \theta_{-t}(\theta_T\omega), C(\theta_{-t}(\theta_T\omega))) \\ &= \cap_{t \geq 0} G(t, \theta_{-t}(\theta_T\omega), C(\theta_{-t}(\theta_T\omega))) \\ &= A(\theta_T\omega). \end{aligned}$$

proving that $A(\theta_T\omega) \subset G(T, \omega)A(\omega)$. But since A is already forward invariant, this proves the invariance of A under G .

Next, by Lemma 3.1.1. (vi), if C is a random compact set, so are $\omega \mapsto C(\theta_{-t}\omega)$ and $\omega \mapsto G(t, \theta_{-t}\omega)C(\theta_{-t}\omega)$. Finally, replace the decreasing intersection over \mathbb{R}_+ in (8) by the countable intersection over \mathbb{N} , and use Lemma 3.1.1.(iv) to conclude that A is a random compact set. \square

DEFINITION.— (i) A is a **global attractor** of G if, for any r.v. X ,

$$\mathbf{P} \lim_{t \rightarrow \infty} d(G(t, \cdot)X(\cdot), A(\rho_t \cdot)) = 0.$$

(ii) A is **globally asymptotically stable** if it is stable and a global attractor.

Lastly, since we want to extend Lyapounov's second method to our set-valued random dynamical system G , we need an appropriate generalization of a Lyapounov function. We follow Arnold & Schmalfuss [2]. Let A be a G -invariant random set.

DEFINITION.— $\mathcal{V} : \Omega \times \tau \rightarrow \mathbb{R}_+$ is a **Lyapounov function** for A under G if:

- (i) $\mathcal{V}(\cdot, x)$ is measurable $\forall x \in \tau$, and $\mathcal{V}(\omega, \cdot)$ is continuous $\forall \omega \in \Omega$.
- (ii) \mathcal{V} is uniformly unbounded, i.e., $\lim_{\|x\| \rightarrow \infty} \mathcal{V}(\omega, x) = +\infty \forall \omega$.
- (iii) $\mathcal{V}(\omega, x) = 0$ for $x \in A(\omega)$ and $\mathcal{V}(\omega, x) > 0$ for $x \notin A(\omega)$.
- (iv) \mathcal{V} is strictly decreasing along orbits of G not in $A(\omega)$:
 $\forall v \in \mathcal{V}(t, \omega, \varphi(t, \omega, x)), \forall w \in \mathcal{V}(\omega, x),$

$$v < w \text{ as long as } x \notin A(\omega).$$

THEOREM 2.— *Let G be an upper-semi continuous random multivalued dynamical system taking non-empty, compact and convex values, A a random compact set that is invariant with respect to G , and suppose there exists a Lyapounov function \mathcal{V} . Then, A is globally asymptotically stable.*

Proof of Theorem 2. We first need the analog of Lemmata 6.3. and 6.4 in Arnold & Schmalfuss [2]. The proofs are exactly the same as in Arnold & Schmalfuss [2] and are not repeated here.

LEMMA 3.1.3.— *Let \mathcal{V} be a Lyapounov function for A . Then, for any $\delta > 0$,*

$$C_\delta(\omega) := \overline{V^{-1}(\omega, [0, \delta])}$$

is a forward invariant random compact set.

LEMMA 3.1.4.— *Let \mathcal{V} be a Lyapounov function for A and G . Denote for any fixed $\varepsilon > 0$ by $B_\varepsilon(A(\omega)) := \{x : d(x, A(\omega)) \leq \varepsilon\}$. Then, for any $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta > 0$ such that*

- (i) $\mathbf{P}[\overline{\mathcal{V}^{-1}(\cdot, [0, \delta])} \subset B_\varepsilon(A(\cdot))] \geq 1 - \eta,$
- (ii) $\mathbf{P}[X(\cdot) \in B_\varepsilon(A(\cdot))] \geq 1 - \eta,$ *for any random variable $X(\cdot)$ satisfying $\mathbf{P}[\mathcal{V}(\cdot, X(\cdot)) \leq \frac{\delta}{2}] \geq 1 - \frac{\eta}{2}.$*

Notice that, in the last theorem, δ is not a random variable but a fixed real number.

Back to the proof of Theorem 2, we first show that A is stable. For this purpose, take an arbitrary $\varepsilon > 0$, and choose $\delta > 0$ according to Lemma 3.1.4 with $\eta = \varepsilon$. Lemma 3.1.3 implies that

$$C_\delta(\omega) := \overline{\mathcal{V}^{-1}(\omega, [0, \delta])}$$

is a forward invariant random compact set. Moreover, it is clearly a neighborhood of A . It follows from the above choice of δ that $\mathbf{P}[d(C_\delta|A) \geq \varepsilon] \leq \varepsilon$. Hence, A is stable.

It remains to prove that A is an attractor. For this purpose, we need a last Lemma, the proof of which is entirely analogous to that of Prop. 6.2. in Arnold & Schmalfuss [2].

LEMMA 3.1.5.— *If there exists a Lyapounov function for A , then any other invariant random compact set A' satisfies $A'(\omega) \subset A(\omega)$ on a θ -invariant set of states ω of full \mathbf{P} -measure.*

Let X be any random variable. Since \mathcal{V} is uniformly unbounded, we can choose for any $\varepsilon > 0$, some $N > 0$ sufficiently large so that $\mathbf{P}[X \in C_N] \geq 1 - \frac{\varepsilon}{2}$. By Lemma 3.1.3, C_N is a forward invariant random compact under G , which is also a neighborhood of A . By Lemma 3.1.4.,

$$A'_N(\omega) := \cap_{t \geq 0} G(t, \theta_{-t}\omega) C_N(\theta_t\omega)$$

is an invariant random compact set. Moreover A'_N is attracting C_N in the “pull-back” sense, which implies that for all $t \geq T(\varepsilon, N)$

$$\mathbf{P}[d(G(t, \cdot, C_N(\cdot)))|A'_N(\theta_t \cdot)] < \frac{\varepsilon}{2}.$$

Since $A(\omega) \subset A'_N$, Lemma 3.1.5 entails that $A = A'_N$ for any N on an invariant subset of Ω of full measure \mathbf{P} . But

$$\begin{aligned} d(G(t, \omega, X(\omega)), A(\theta_t\omega)) \\ \leq d(G(t, \omega, X(\omega)), G(t, \omega, C_N(\omega))) + d(G(t, \omega, C_N(\omega))|A(\theta_t\omega)). \end{aligned}$$

Hence, using the fact that $X(\omega) \in C_N(\omega)$ iff $G(t, \omega, X(\omega)) \subset G(t, \omega, C_N(\omega))$ for all $t \geq 0$, it follows that

$$\mathbf{P} - \lim_{t \rightarrow +\infty} d(G(t, \cdot)X(\cdot), A(\theta_t \cdot)) = 0.$$

Consequently, A is attracting. □

Step 2.

Observe that, in the perfectly competitive case,

$$G(t, \omega)x = x + \int_0^t \Phi(\bar{s}(x(\tau), \rho_\tau)) d\tau \quad \omega \in \Omega \quad (10)$$

is an upper-semi continuous set-valued random semi-flow with non-empty, convex and compact values, where Φ is Filippov's correspondence associated to φ . Moreover, the function:

$$\mathcal{V}(z, \omega, t) := \sum_i \nabla V_i(z_i, \omega, t) \cdot \dot{z}_i(\omega, t),$$

is a stochastic Lyapounov function in the sense given to this word in this proof (see Giraud [20] for the details in the deterministic case). The unique property that is not an obvious consequence of our standing hypotheses is the uniform unboundedness of \mathcal{V} . However, since this latter function is only defined on the compact subset τ , it suffices to extend it in a continuous (wrt z) and measurable (wrt ω) way out of τ , and to choose an extension that explodes to infinity as $\|z\| \rightarrow +\infty$. One therefore gets:

COROLLARY.— *Under (C), when competition is perfect, the random set of constrained efficient financial allocations is globally asymptotically stable for $T = +\infty$.*

3.2 Proof of Prop. 2.

Suppose I is finite. Take a trade curve φ such that, at no point of time t , players coordinate on the no-trade NE of $G[T_{z,\omega,t}\mathcal{E}]$ except if z is constrained efficient at time t . According to Lemma 1, therefore, the direction followed by φ on each z is some non-autarkic NE. Such a NE outcome \dot{z} involving non-trivial trades is individually rational with respect to short-run preferences:

$$\nabla V_i(z_i(t), \omega, t) \cdot \dot{z}_i > 0 \quad \forall i.$$

Indeed, every player can defend $\dot{z}_i = 0$ by bidding 0 on each commodity. It follows that \mathcal{V} as defined above is still a Lyapounov function. Thus, the conclusion of Step 2 applies.

Finally, if some trade curve admits a time t for which players coordinate on some no-trade NE in $G[T_{z,\omega,t}\mathcal{E}]$, and if z is not constrained efficient, we claim that z is locally unstable in the sense of Definition 4. Indeed, take any random neighborhood V of z sufficiently small so that, for \mathbf{P} -a.e. ω , $V(\omega)$ has an empty intersection with the closed subset of constrained efficient allocations. For each ω , choose some point $x(\omega) \in V(\omega)$ which is such that:

$$V_i(x_i, t) > V_i(z_i, t) \quad \text{for every } i.$$

Such a point x exists because, by assumption, z is not second-best efficient. Under (C), the linear economy $T_{x,t}\mathcal{E}$ admits some Walras equilibrium, which does not reduce to no-trade (because x is not efficient) and does not point in the direction of z . Now, according to Lemma 1, $G[T_{x,t}\mathcal{E}]$ admits this Walras equilibrium as NE. Continuing the argument, one constructs a path $\varphi : [0, \varepsilon) \rightarrow \tau$, solution to (6), with $\varphi(0) = x$, and such that, for every $\varepsilon > t \geq 0$, $\dot{\varphi}$ points in the direction of some non-trivial Walras equilibrium of $T_{\varphi(t)}\mathcal{E}$. Apply Theorem 1 to φ , and conclude that, for a.e. ω , there must be some time t_ω^* such that $\varphi(t_\omega^*) \notin C_\omega$. Hence the conclusion.

3.3 Proof of Prop. 3.

The two inclusions follow from Lemma 1 and from the analogous result obtained in Weyers [26], [27] for the NE of the one-shot strategic market game with limit-prices. We can rephrase the results proven there in our setting as follows. Fix ω . As soon as z is not constrained efficient, the set of NE outcomes of the game $G_k[T_z\mathcal{E}]$ with k replicas for each player i strictly contains the set of NE outcomes of $G_{k+1}[T_z\mathcal{E}]$. And this latter set also strictly contains the short-term outcome of $T_z\mathcal{E}$, as shown in Lemma 1.

3.4 Proof of Prop. 4.

According to Weyers ([27], Prop. 5), when there are two types of agents and two assets with interior initial endowments, the set of pure Nash equilibrium allocations coincides with the competitive offer area (see Weyers ([27], Definition 8). Applied to a linear tangent economy, this results says that the subset of directions in which an economy can move on from z equals the subset of directions given by the competitive offer set of its tangent economy $T_z\mathcal{E}$, hence with the set of directions that are individually rational for both players. (Indeed, the competitive offer curve of each agent i is tangent to her indifference line at z_i .) The conclusion therefore follows from Schecter [25].

4 Concluding remarks

We end this paper with a few remarks.

a) Needless to say, the notion of convergence used here is very weak. We suspect that the convergence result fails whenever one asks for more stringent notions of convergence (such as convergence almost surely or in L^p). We leave for further research the task of finding a counterexample for stronger notions of convergence.

b) In this paper, information was assumed to be incomplete but public: all the households are simultaneously informed through the information filtration $(\mathcal{F}_t)_t$. One open question is how to extend this work to a set-up with asymmetric information. For this purpose, Cornet & de Boisdeffre [8] could be instrumental.

c) For simplicity, we restricted ourselves to economies with finitely many types even in the perfectly competitive case. Using the same trick as in Giraud [20], one could nevertheless perform the same analysis of the perfectly competitive case for large economies with a finite number of types of *preferences* but otherwise arbitrary (integrable) initial endowments of assets $z(0)$.

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